Why does $\frac{d}{dx}e^x = e^x$?

Suppose we want to find the number a such that $\frac{d}{dx}a^x = a^x$. Then, using the limit definition of derivative,

$$\frac{d}{dx}a^x = \lim_{h\to 0}\frac{a^{x+h}-a^x}{h} = \lim_{h\to 0}\frac{a^x a^h - a^x}{h} = a^x \lim_{h\to 0}\frac{a^h-1}{h}$$

Now, substituting, $n = a^h - 1$, $n+1 = a^h$, $h = \log_a n+1$

$$\frac{d}{dx}a^x = a^x \lim_{n\to 0}\frac{n}{\log_a(n+1)} = e^x \lim_{n\to 0}\frac{1}{\frac{1}{n}\log_a(n+1)}$$

$$= a^x \lim_{n\to 0}\frac{1}{\log_a\left((n+1)^{\frac{1}{n}}\right)} = a^x \frac{1}{\log_a\left(\lim_{n\to 0}\left((n+1)^{\frac{1}{n}}\right)\right)}$$

Now, it is clear that only if $a = \lim_{n\to 0}\left((n+1)^{\frac{1}{n}}\right)$ we have

$$\frac{d}{dx}a^x = a^x \frac{1}{\log_a\left(\lim_{n\to 0}(n+1)^{\frac{1}{n}}\right)} = a^x \frac{1}{\log_a(a)} = a^x \frac{1}{1} = a^x$$

Thus $\frac{d}{dx}a^x = a^x$ implies $a = \lim_{n\to 0}(n+1)^{\frac{1}{n}}$, which

is why $e \equiv \lim_{n\to 0}(1+n)^{\frac{1}{n}}$